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A Note on Defining an Extension of a Probability Measure on Subsets of Function Space, By Applying One of J. L. Doob's Theorems

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A Note on Defining an Extension of a Probability Measure on Subsets of Function Space, By Applying One of J. L. Doob's Theorems*

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The following definitions are based on concepts in references (2) and (4).

Definition 1: Let W denote the set of all real valued functions, w(t), of a real variable t.

Definition 2: Let $t_1, \ldots, t_n$ be a finite set of t values. Let $-\infty \leqq a_i < b_i \leqq +\infty$, $i = 1, \ldots, n$. Then the set N of functions:

$$N = \underset{w(t)\,\epsilon\, W}{E} \left\{ a_i < w(t_i) < b_i, i=1, \ldots, n \right\}$$

is defined to be a *neighborhood* of any function in the set N.

It is noted that W is itself a neighborhood of each function in W.

Definition 3: An *open set* in W is any union of neighborhoods.

Definition 4: A Borel Field F of subsets of W is a class of subsets of W such that $W \epsilon F$, and whenever $A \epsilon F$ and $B \epsilon F$, then $A - B \epsilon F$; and whenever each set of the sequence of sets $A_1, \ldots, A_n, \ldots$, is in F, then so is the union of the sequence.

It is noted that the intersection of the sets in the sequence is also in F.

Definition 5: A probability measure on subsets of function space is a non-negative, completely-additive, complete, set function P(A) defined on a Borel Field of subsets of W, and such that $P(W) = 1$.

Definition 6: F_2 is the Borel field generated by the open sets.

Definition 7: F_0 is the Borel field generated by the neighborhoods.

Definition 8: Let P be any probability measure defined on a Borel field F of subsets of W; then the three concepts W,F,P, together, are defined to be a *Borel Probability Field in Function Space,* abbreviated by bpf, and denoted by (W,F,P).

Definition 9: A bpf (W,F_0,P_0) is called a *Fundamental Borel Probability Field in Function Space,* appreviated fbpf.

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Definition 10: If (W,F,P) is any bpf, and if $A \leq W$, then

$$P^*(A) = \inf_{E} P(E),\ E \epsilon F,\ E \geq A,$$ is defined to be the *outer P measure of the set A.*

Definition 11: Let (W,F,P) be a bpf, and let W' be a subset of W such that $P^*(W') = 1$. Let $\widetilde{W'}$ denote the complement of W'. If A is any subset of W such that

$A = EW' + H\widetilde{W'}$, where E and H belong to F, then define the set function $P'(A)$ by:

$P'(A) = P(E)$. It can be shown that the class F' of all sets of the form $EW' + H\widetilde{W'}$, where E and H belong to F, is a Borel field which includes F, and that $P'(A)$ is a probability measure defined on a Borel field which includes F', and such that P' reduces to P on the sets in F. Then the bpf (W,F',P') is called an *adjunction extension* of (W,F,P) and is said to be obtained from (W,F,P) by adjoining W' to F.

The following theorem is a formalization of statements made by J. L. Doob (2:p.23; 3:p.69) and is also based on a theorem of J. L. Doob's (1,p.109, theorem 1.1):

Theorem I: Let (W,F,P) be an arbitrary bpf. A necessary and sufficient condition that there exist an adjunction extension (W,F',P') of (W,F,P) obtained by adjoining W' to F is that $P^*(W') = 1$.

Proof: This theorem is proved by showing that $P'(E)$, as defined in the statement of the theorem is a probability measure (unique up to sets of P measure zero) such that its domain of definition includes the Borel field $F' \geq F$, and such that P' reduces to P on F.

The complete additivity and other probability measure properties of the P' measure follow from the corresponding properties of P measure, and from the uniqueness of P' which itself follows from Doob's theorem (1,p.109, theorem 1.1). The fact that F' is a Borel field which includes F follows from the properties of F.

Now following Doob and S. Kakutani (2,p.25) define the set function P_2^* on the subsets of W as follows:

Definition 12: If G is any open set in W, and (W,F_0,P_0) is any fbpf, then let $P_{2*}(G) = \sup_{E_0} P_0(E_0),\ E_0 \epsilon F_0,\ E_0 \leq G.$

Definition 13: If A is any subset of W, then let

$$(A) = \inf_{G} P_{2*}(G),\ G \text{ open},\ G \geq A$$

It can then be shown that P_2^* is an outer measure, and that the P_2^* measurable sets include the Borel field F_2.

Let P_2 denote the P_2^* measure of the sets in F_2. P_2 measure is called *Kakutani measure.*

It is sometimes desirable to know whether, when there exists an adjunction extension (W,F'_0,P'_0) of (W,F_0,P_0), the adjoined set W' belongs to F_2 (2,p.29); the reason for this being that it is desirable to use the bpf (W,F_2,P_2) in studying probabilities in function space. (2,p.25,26,29). It is clear that an adjunction extension (W,F'_2,P'_2) of (W,F_2,P_2), if it exists, corresponding to the adjunction extension (W,F'_0,P'_0) of (W,F_0,P_0) would serve the same purpose, even though W' might not belong to F_2. The condition for the existence of this extension is given in the following theorem, which is Theorem I applied to the bpf (W,F_2,P_2).

Theorem II: Suppose (W,F'_0,P'_0) is an adjunction extension of a fbpf obtained by adjoining W' to F_0. Then a necessary and sufficient condition that there exist a corresponding adjunction extension of (W,F_2,P_2) obtained by adjoining W' to F_2 is that $P_2^*(W') = 1$.

Proof: The proof is the same as in Theorem I, except that (W,F_2,P_2) is used instead of the arbitrary bpf.

Without going into the concept of a measurable Borel Probability Field in function space (2,p.26-29), it may be stated that whenever the condition of Theorem II is satisfied for a measurable adjunction extension of a fbpf, then the corresponding adjunction extension of (W,F_2,P_2) is also measurable. This follows from the fact that $F'_0 \leq F'_2$.

Literature Cited

1. J. L. Doob. Stochastic Processes Depending on a Continuous Parameter: Transactions of the American Mathematical Society, Vol. 42, No. 1, 1937; pp. 107-140.
2. ———. Probability in Function Space: Bulletin of the American Mathematical Society, Vol. 53, Part 1, Jan. 1947; pp. 15-30.
3. ———. Stochastic Processes, John Wiley and Sons, Inc., New York, 1953.
4. A. N. Kolmogorov. Foundations of Probability, Second English Edition, Chelsea Publishing Co., New York, 1956.

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